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Theory and Methodology

Applications of the extent analysis method on fuzzy AHP

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Abstract

In this paper, a new approach for handling fuzzy AHP is introduced, with the use of triangular fuzzy numbers for pairwise comprison scale of fuzzy AHP, and the use of the extent analysis method for the synthetic extent value S_i of the pairwise comparison. By applying the principle of the comparison of fuzzy numbers, that is, $V(M_1 \geqslant M_2) = 1$ iff $m_1 \geqslant m_2$, $V(M_2 \geqslant M_1) = \text{hgt}(M_1 \cap M_2) = \mu_{M_1}(d)$, the vectors of weight with respect to each element under a certain criterion are represented by $d(A_i) = \min V(S_i \geqslant S_k)$, $k = 1, 2, \ldots, n$; $k \neq i$. This decision process is demonstrated by an example.

Keywords: Fuzzy AHP; Triangular fuzzy mumbers; Extent analysis; Comparison of fuzzy numbers

1. Introduction

Many scholars have engaged in the fuzzy extension of Saaty's priority theory. Since the publication of Saaty's *The Analytic Hierarchy Process* (for short AHP), Netherlands's scholars van Laarhoven and Pedrycg [3] proposed a method, where the fuzzy comparing judgment is represented by triangular fuzzy numbers. They used fuzzy numbers with triangular membership function and simple operation laws. According to the method of logarithmic least squares (for short LLMS), the priority vectors were obtained.

In this paper, a new approach to handling fuzzy AHP is given, which is different from the above-mentioned methods. But the ordering of a permutation with respect to elements is quite the same. First of all, triangular fuzzy numbers are used for a pairwise comparison scale of fuzzy AHP. Then, by using the extent analysis method [1], the synthetic extent value S_i of the pairwise comparison is introduced, and by applying the principle of the comparison of fuzzy numbers [1],

$$V(M_1 \geqslant M_2) = 1 \quad \text{iff } m_1 \geqslant m_2$$

and

$$V(M_2 \geqslant M_1) = \text{hgt}(M_1 \cap M_2) = \mu_{M_1}(d),$$

the weight vectors with respect to each element under a certain criterion can be represented by

$$d(A_i) = \min V(S_i \geqslant S_k), \quad k = 1, \ldots, n, \ k \neq i.$$

Finally, an example is given to explain this decision process.

2. Basic concept of fuzzy AHP

2.1. Triangular fuzzy numbers

Definition 1. Let $M \in F(R)$ be called a fuzzy number if:

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1) exists $x_0 \in R$ such that $\mu_M(x_0) = 1$.
2) For any $\alpha \in [0, 1]$,

$$A_\alpha = \left[x, \mu_{A_\alpha}(x) \geqslant a\right]$$

is a closed interval.Here $F(R)$ represents all fuzzy sets, and R is the set of real numbers.

Definition 2. We define a fuzzy number M on R to be a triangular fuzzy number if its membership function $\mu_M(x): R \to [0, 1]$ is equal to

$$\mu_M(x) = \begin{cases} \dfrac{x}{m-l} - \dfrac{l}{m-l}, & x \in [l, m], \\ \dfrac{x}{m-u} - \dfrac{u}{m-u}, & x \in [m, u], \\ 0, & \text{otherwise}, \end{cases} \quad (1)$$

where $l \leqslant m \leqslant u$, l and u stand for the lower and upper value of the support of M respectively, and m for the modal value. The triangular fuzzy number can be denoted by (l, m, u). The support of M is the set of elements $\{x \in R \mid l < x < u\}$. When $l = m = u$, it is a nonfuzzy number by convention.

Consider two triangular fuzzy numbers M_1 and M_2, $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$. Their operational laws are as follows:

1. $(l_1, m_1, u_1) \oplus (l_2, m_2, u_2)$
$= (l_1 + l_2, m_1 + m_2, u_1 + u_2)$. (2)
2. $(l_1, m_1, u_1) \odot (l_2, m_2, u_2)$
$\approx (l_1 l_2, m_1 m_2, u_1 u_2)$. (3)
3. $(\lambda, \lambda, \lambda) \odot (l_1, m_1, u_1) = (\lambda l_1, \lambda m_1, \lambda u_1)$,
$\lambda > 0, \lambda \in R$. (4)
4. $(l_1, m_1, u_1)^{-1} \approx (1/u_1, 1/m_1, 1/l_1)$. (5)

2.2. Value of fuzzy synthetic extent

Let $X = \{x_1, x_2, \ldots x_n\}$ be an object set, and $U = \{u_1, u_2, \ldots, u_m\}$ be a goal set. According to the method of extent analysis [1], we now take each object and perform extent analysis for each goal respectively. Therefore, we can get m extent analysis values for each object, with the following signs:

$$M_{g_i}^1, M_{g_i}^2, \ldots, M_{g_i}^m, \quad i = 1, 2, \ldots, n, \quad (6)$$

where all the $M_{g_i}^j$ $(j = 1, 2, \ldots, m)$ are triangular fuzzy numbers.

Definition 3. Let $M_{g_i}^1, M_{g_i}^2, \ldots, M_{g_i}^m$ be values of extent analysis of ith object for m goals. Then the value of fuzzy synthetic extent with respect to the i-th object is defined as [1]

$$S_i = \sum_{j=1}^{m} M_{g_i}^j \odot \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^j\right]^{-1}. \quad (7)$$

3. Presentation method of fuzzy numbers for the pairwise comparison scale

The first task of the fuzzy AHP method is to decide on the relative importance of each pair of factors in the same hierarchy. By using triangular fuzzy numbers, via pairwise comparison, the fuzzy evaluation matrix $A = (a_{ij})_{n \times m}$ is constructed. For example, essential or strong importance of element i over element j under a certain criterion: then $a_{ij} = (l, 5, u)$, where l and u represent a fuzzy degree of judgment. The greater $u - l$, the fuzzier the degree; when $u - l = 0$, the judgment is a nonfuzzy number. This stays the same to scale 5 under general meaning. If strong importance of element j over element i holds, then the pairwise comparison scale can be represented by the fuzzy number

$$a_{ij}^{-1} = (1/u, 1/m, 1/l).$$

4. Calculation of priority vectors of the fuzzy AHP

Let $A = (a_{ij})_{n \times m}$ be a fuzzy pairwise comparison matrix, where $a_{ij} = (l_{ij}, m_{ij}, u_{ij})$, which are satisfied with

$$l_{ij} = \frac{1}{l_{ji}} = \quad m_{ij} = \frac{1}{m_{ji}}, \quad u_{ij} = \frac{1}{u_{ji}}.$$

To obtain the estimates for the vectors of weights under each criterion, we need to consider a principle of comparison for fuzzy numbers. In fact, two questions may arise.

1) What is the fuzzy value of the least or greatest number from a family of fuzzy numbers?

2) Which is the greatest or the least among several fuzzy numbers?

The answer to the first question is given by the use of the operation max and min [2]. However, the answer to the second question requires efforts. We must evaluate the degree of possibility for $x \in R$ fuzzily restricted to belong to M, to be greater than $y \in R$ fuzzily restricted to belong to M. Thus, we give the definition as follows:

Definition 4. The degree of possibility of $M_1 \geqslant M_2$ is defined as

$$V(M_1 \geqslant M_2) = \sup_{x \geqslant y} \left[\min\left(\mu_{M_1}(x), \mu_{M_2}(y)\right)\right]. \tag{8}$$

When a pair (x, y) exists such that $x \geqslant y$ and $\mu_{M_1}(x) = \mu_{M_2}(y) = 1$, then we have $V(M_1 \geqslant M_2) = 1$. Since M_1 and M_2 are convex fuzzy numbers we have that

$$V(M_1 \geqslant M_2) = 1 \quad \text{iff} m_1 \geqslant m_2,$$

$$V(M_2 \geqslant M_1) = \text{hgt}(M_1 \cap M_2) = \mu_{M_1}(d), \tag{9}$$

where d is the ordinate of the highest intersection point D between μ_{M_1} and μ_{M_2} (see Fig. 1).

When $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$, the ordinate of D is give by Eq. (10).

$$V(M_2 \geqslant M_1) = \text{hgt}(M_1 \cap M_2) = \frac{l_1 - u_2}{(m_2 - u_2) - (m_1 - l_1)}. \tag{10}$$

To compare M_1 and M_2, we need both the values of $V(M_1 \geqslant M_2)$ and $V(M_2 \geqslant M_1)$.

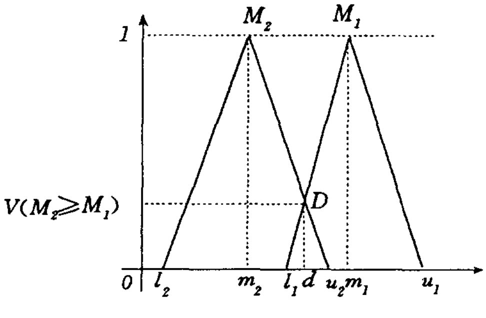

Fig. 1.

Table 1
The matrix $\mathscr{R}$, pairwise comparison of performance criteria

	C_1	C_2	C_3	C_4
C_1	$(1, 1, 1)$	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$ $(\frac{3}{2}, 2, \frac{5}{2})$	$(\frac{2}{3}, 1, \frac{3}{2})$	$(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$ $(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$
C_2	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{3}{2}, 2, \frac{5}{2})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$	$(1, 1, 1)$	$(\frac{5}{2}, 3, \frac{7}{2})$ $(\frac{5}{2}, 3, \frac{7}{2})$	$(\frac{2}{5}, 1, \frac{3}{2})$ $(\frac{2}{5}, 1, \frac{3}{2})$ $(\frac{2}{3}, 2, \frac{5}{2})$
C_3	$(\frac{2}{3}, 1, \frac{3}{2})$	$(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$ $(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$	$(1, 1, 1)$	$(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$
C_4	$(\frac{5}{2}, 3, \frac{7}{2})$ $(\frac{5}{2}, 3, \frac{7}{2})$ $(\frac{3}{2}, 2, \frac{5}{2})$	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$	$(\frac{3}{2}, 2, \frac{5}{2})$	$(1, 1, 1)$

Definition 5. The degree possibility for a convex fuzzy number to be greater than k convex fuzzy numbers M_i $(i = 1, 2, \ldots, k)$ can be defined by

$$\begin{aligned} V(M \geqslant M_1, M_2, \ldots, M_k) &= V\left[(M \geqslant M_1) \text{ and } (M \geqslant M_2) \right. \\ &\qquad \left. \text{and } \cdots \text{ and}(M \geqslant M_k)\right] \\ &= \min V(M \geqslant M_i), \quad i = 1, 2, \ldots, k. \end{aligned} \tag{11}$$

Assume that

$$d'(A_i) = \min V(S_i \geqslant S_k), \tag{12}$$

for $k = 1, 2, \ldots, n$; $k \neq i$. Then the weight vector is given by

$$W' = \left(d'(A_1), d'(A_2), \ldots, d'(A_n)\right)^{\mathrm{T}}, \tag{13}$$

where A_i $(i = 1, 2, \ldots, n)$ are n elements.

Via normalization, we get the normalized weight vectors

$$W = \left(d(A_1), d(A_2), \ldots, d(A_n)\right)^{\mathrm{T}}. \tag{14}$$

where W is a nonfuzzy number.

5. Application of fuzzy AHP in group decisions

The following example is a modification of the problem originally presented by van Laarhoven [2]. Suppose that at a university the post of a professor in

Table 2

	C_1	C_2	C_3	C_4	W_C
C_1	(1, 1, 1)	(0.86, 1.17, 1.56)	(0.67, 1, 1.5)	(0.33, 0.39, 0.49)	0.13
C_2	(0.64, 0.85, 1.16)	(1, 1, 1)	(2.5, 3, 3.5)	(0.95, 1.33, 1.83)	0.41
C_3	(0.87, 1, 1.49)	(0.29, 0.33, 0.40)	(1, 1, 1)	(0.4, 0.5, 0.67)	0.03
C_4	(2.04, 2.56, 3.03)	(0.55, 0.75, 1.05)	(1.49, 2, 2.5)	(1, 1, 1)	0.43

Operations Research is vacant, and three serious candidates remain. We shall call them A_1, A_2 and A_3. A committee has convened to decide which applicant is best qualified for the job. The committee has three members and they have identified the following decision criteria:

1) mathematical creativity (C_1);
2) creativity implementations (C_2);
3) administrative capabilities (C_3);
4) human maturity (C_4).

First step. Via pairwise comparison, the fuzzy evaluation matrix $\mathscr{R}$, which is relevant to the objective, is constructed (see Table 1).

By using formula (2) and taking the average value, we obtain Table 2.

Then, by applying formula (7), we have

$$S_1 = (2.86, 3.56, 4.55) \odot \left(\frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right)$$
$$= (0.12, 0.19, 0.29),$$

$$S_2 = (5.09, 6.18, 7.49) \odot \left(\frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right)$$
$$= (0.22, 0.32, 0.48),$$

$$S_3 = (2.56, 2.83, 3.56) \odot \left(\frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right)$$
$$= (0.11, 0.15, 0.23),$$

$$S_4 = (5.08, 6.31, 7.58) \odot \left(\frac{1}{23.18}, \frac{1}{18.88}, \frac{1}{15.59} \right)$$
$$= (0.21, 0.33, 0.49).$$

Using formulas (9) and (10),

$$V(S_1 \geqslant S_2) = \frac{0.22 - 0.29}{(0.19 - 0.29) - (0.32 - 0.22)} = 0.35,$$

$$V(S_1 \geqslant S_3) = 1,$$

$$V(S_1 \geqslant S_4) = \frac{0.21 - 0.29}{(0.19 - 0.29) - (0.33 - 0.21)} = 0.32,$$

$$V(S_2 \geqslant S_1) = 1 \quad V(S_2 \geqslant S_3) = 1,$$

$$V(S_2 \geqslant S_4) = \frac{0.21 - 0.48}{(0.32 - 0.48) - (0.33 - 0.21)} = 0.96,$$

$$V(S_3 \geqslant S_1) = 0.73,$$
$$V(S_3 \geqslant S_2) = 0.06,$$
$$V(S_3 \geqslant S_4) = 0.10,$$
$$V(S_4 \geqslant S_1) = 1,$$
$$V(S_4 \geqslant S_2) = 1,$$
$$V(S_4 \geqslant S_3) = 1.$$

Finally, by using formula (12), we obtain

$$d'(C_1) = V(S_1 \geqslant S_2, S_3, S_4) = \min(0.35, 1, 0.32) = 0.32,$$

Table 3a
The matrix $\mathscr{R}_1$

C_1	A_1	A_2	A_3
A_1	(1, 1, 1)	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{3}, 1, \frac{3}{2})$	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$
A_2	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{3}, 1, \frac{3}{2})$	(1, 1, 1)	$(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$
A_3	$(\frac{2}{3}, 1, \frac{3}{2})$ $(\frac{2}{3}, 2, \frac{5}{2})$	$(\frac{2}{3}, 2, \frac{5}{2})$	(1, 1, 1)

Table 3b
The matrix $\mathscr{R}_2$

C_2	A_1	A_2	A_3
A_1	(1, 1, 1)	$(\frac{5}{2}, 3, \frac{7}{2})$	$(\frac{3}{2}, 2, \frac{5}{2})$
A_2	$(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$	(1, 1, 1)	–
A_3	$(\frac{5}{2}, \frac{1}{2}, \frac{2}{3})$	–	(1, 1, 1)

Table 3c
The matrix $\mathscr{R}_3$

C_3	A_1	A_2	A_3
A_1	(1, 1, 1)	$(\frac{5}{2}, 3, \frac{7}{2})$ $(\frac{5}{2}, 3, \frac{7}{2})$ $(\frac{3}{2}, 2, \frac{5}{2})$	$(\frac{5}{2}, 3, \frac{7}{2})$
A_2	$(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$ $(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$ $(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$	(1, 1, 1)	$(\frac{2}{3}, 1, \frac{3}{2})$
A_3	$(\frac{2}{7}, \frac{1}{3}, \frac{2}{5})$	$(\frac{2}{3}, 1, \frac{3}{2})$	(1, 1, 1)

$$d'(C_2) = V(S_2 \geq S_1, S_3, S_4)$$
$$= \min(1, 1, 0.96) = 0.96,$$
$$d'(C_3) = V(S_3 \geq S_1, S_2, S_4)$$
$$\min(0.73, 0.06, 0.10) = 0.06,$$
$$d'(C_4) = V(S_4 \geq S_1, S_2, S_3)$$
$$= \min(1, 1, 1) = 1.$$

Therefore,

$$W' = (0.32, 0.96, 0.06, 1)^{T}$$

Table 3d
The matrix $\mathscr{R}_4$

C_4	A_1	A_2	A_3
A_1	(1, 1, 1)	–	$(\frac{3}{2}, 2, \frac{2}{5})$ $(\frac{5}{2}, \frac{1}{2}, \frac{2}{5})$
A_2	–	(1, 1, 1)	$(\frac{3}{2}, 2, \frac{2}{5})$
A_3	$(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$ $(\frac{3}{2}, 2, \frac{5}{2})$	$(\frac{2}{5}, \frac{1}{2}, \frac{2}{3})$	(1, 1, 1)

via normalization, and we have obtained the weight vectors with respect to the decision criteria C_1, C_2, C_3 and C_4:

$$W = (0.13, 0.41, 0.03, 0.43)^{T}.$$

Second step. At the second level of the decision procedure, the committee compares candidates A_1, A_2 and A_3 under each of the criteria separately. This results in the matrices $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ and $\mathscr{R}_4$, which are shown in Tables 3a′–3d′.

In Table 3b, there are two elements such that $l_1 - u_2 > 0$, and in this case, the elements of the matrix must be take normalized.

Table 3a′

C_1	A_1	A_2	A_3	W_{C_1}
A_1	(1, 1, 1)	(0.67, 1, 1.5)	(0.54, 0.75, 1.1)	0.28
A_2	(0.67, 1, 1.5)	(1, 1, 1)	(0.4, 0.5, 0.6)	0.21
A_3	(0.91, 1.33, 1.85)	(1.5, 2, 2.5)	(1, 1, 1)	0.51

Table 3b′

C_2	A_1	A_2	A_3	W_{C_2}
A_1	(0.33, 0.33, 0.34)	(0.28, 0.33, 0.39)	(0.25, 0.33, 0.42)	0.66
A_2	(0.29, 0.33, 0.4)	(0.33, 0.33, 0.34)	–	0.16
A_3	(0.24, 0.32, 0.43)	–	(0.33, 0.33, 0.34)	0.19

Table 3c′

C_3	A_1	A_2	A_3	W_{C_3}
A_1	(0.33, 0.33, 0.34)	(0.27, 0.33, 0.40)	(0.28, 0.33, 0.39)	0.35
A_2	(0.29, 0.32, 0.4)	(0.33, 0.33, 0.34)	(0.21, 0.32, 0.47)	0.33
A_3	(0.28, 0.32, 0.39)	(0.21, 0.32, 0.47)	(0.33, 0.33, 0.34)	0.32

Table 3d′

C_4	A_1	A_2	W_{A_3}	W_{C_4}
A_1	(1, 1, 1)	–	(0.95, 1.25, 1.59)	0.22
A_2	–	(1, 1, 1)	(1.5, 2, 2.5)	0.42
A_3	(0.95, 1.25, 1.59)	(0.4, 0.5, 0.67)	(1, 1, 1)	0.36

As before, these matrices are used to estimate weights, in this case the weights of each candidate under each criterion separately. The results are given in Table 4.

Finally, adding the weights per candidate multiplied by the weights of the corresponding criteria, a final score is obtained for each candidate. Table 5 shows these scores.

The ordering relation between the candidates is exactly the same as in [3]. According to the final scores, it is clear that candidate A_1 is the preferred candidate.

6. Comparison of the extent analysis method and LLSM

According to the complexity of the algorithm, we can distinguish between good and bad points of EAM and LLSM. The time complexity and the space complexity is contained in the complexity of the algorithm.

By time complexity we are refering to the time in which the algorithm was accomplished. We only use the number of times of multiplication, which are more or less as a criterion of appraisal. In this paper, we consider time complexity only.

Assume that we give an $n \times n$ fuzzy pairwise comparison matrix, by using the EAM and LLSM. The weight vectors with respect to each element under certain criterion will then be obtained. Via normalization we get the normalized weight vectors.

Table 4

Criterion	A_1	A_2	A_3
C_1	0.28	0.21	0.51
C_2	0.66	0.16	0.19
C_3	0.35	0.33	0.32
C_4	0.22	0.42	0.36

Let us to count the number of times of multiplication with respect to the two methods, respectively.

Formulas (7), (10) and (14) are major formulas of EAM. In formula (7), to count S_i $(i = 1, 2, \ldots, n)$, we need to use multiplication $6n$ times. In formula (10), via pairwise comparison of $S_1, S_2, \ldots, S_n$, the number of times of multiplication is

$$P_n^2 = n(n-1).$$

Finally, in formula (14), we also need to use multiplication n times.

Therefore, the time complexity of EAM is

$$T_n = 6n + n(n-1) + n = n(n+6). \tag{15}$$

In the LLSM, the normalized weight vectors are

$$w_k = \frac{\left(\prod_{j=1}^{n} a_{kj}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}, \quad k = 1, 2, \ldots, n, \tag{16}$$

where w_k is the k-th component of the weights vector. Evidently, the time complexity of LLSM is

$$T_n' = n[(n+1) + n(n+1) + 1]$$
$$= n(n+1)^2 + n. \tag{17}$$

Thus

$$T_n' - T_n = n(n^2 + n - 4). \tag{18}$$

Let $n = 4$, we obtain $T_4' = 104$, $T_4 = 40$, $T_4' - T_4 = 64$.

Evidently, the EAM is better than the LLSM at time complexity.

Table 5

	A_1	A_2	A_3
Final scores	0.41	0.28	0.25

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